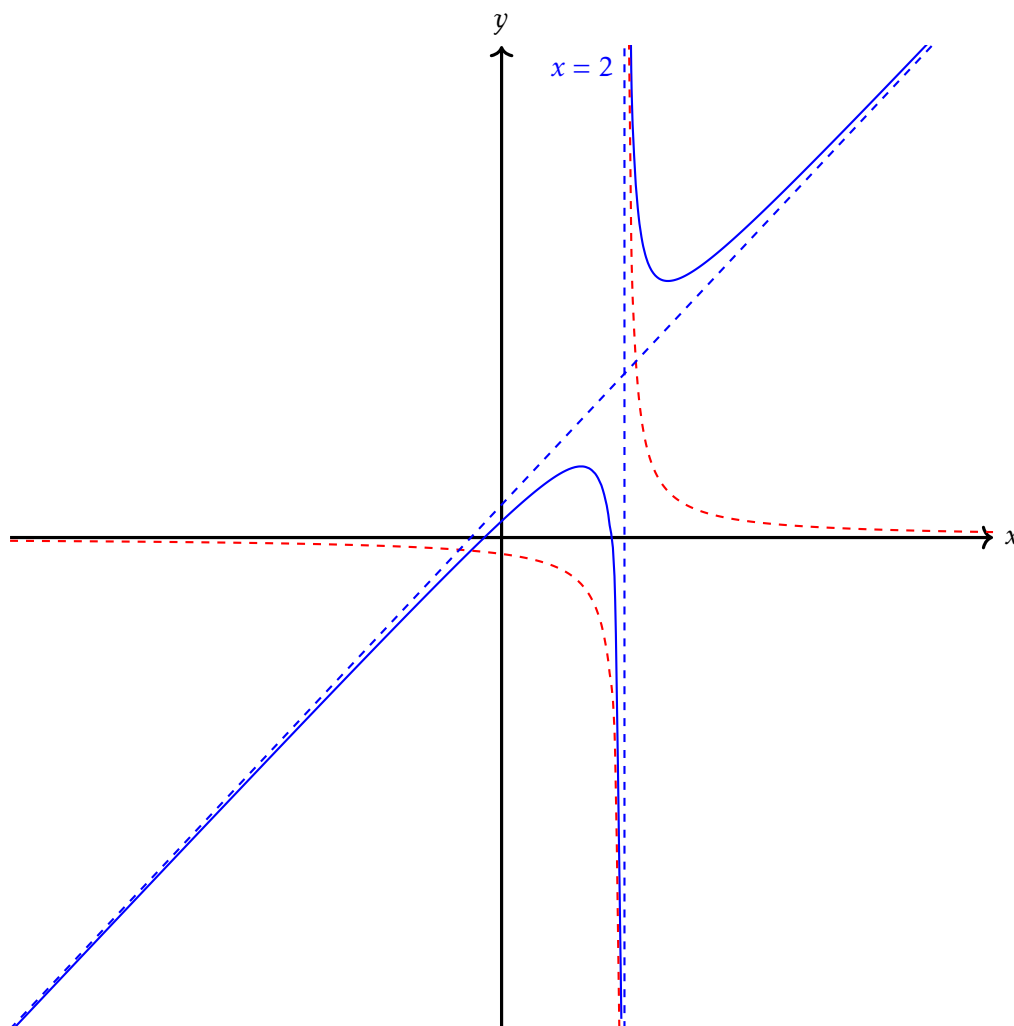


Rational Functions

Example

Sketch $y = \frac{2x^2 - 3x - 1}{x - 2}$

$$\begin{aligned}\frac{2x^2 - 3x - 1}{x - 2} &= \frac{2x(x - 2) + x - 1}{x - 2} \\ &= 2x + \frac{x - 2 + 1}{x - 2} \\ &= 2x + 1 + \frac{1}{x - 2}\end{aligned}$$



Partial Fractions

Example

Write $\frac{1}{(x+5)(x+4)}$ as partial fractions

$$\begin{aligned} & \frac{1}{(x+5)(x+4)} = \frac{A}{x+5} + \frac{B}{x+4} \\ \Rightarrow & 1 = A(x+4) + B(x+5) \\ x = -4 : & 1 = A((-4)+4) + B((-4)+5) \\ & = 0 \cdot A + B \\ \Rightarrow & B = 1 \\ x = -5 : & 1 = A((-5)+4) + B((-5)+5) \\ & = -A + 0 \cdot B \\ \Rightarrow & A = -1 \\ \Rightarrow & \frac{1}{(x+5)(x+4)} = \frac{1}{x+4} - \frac{1}{x+5} \end{aligned}$$

Example

Write $\frac{1}{(x+2)(x-1)^2}$ as partial fractions

$$\begin{aligned} & \frac{1}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{Bx+C}{(x-1)^2} \\ & = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C'}{(x-1)^2} \\ \Rightarrow & 1 = A(x-1)^2 + B(x-1)(x+2) + C'(x+2) \\ x = -2 : & 1 = A((-2)-1)^2 + B((-2)-1)((-2)+2) + C'((-2)+2) \\ & = 9A \\ \Rightarrow & A = \frac{1}{9} \\ x = 1 : & 1 = A(1-1)^2 + B(1-1)(1+2) + C'(1+2) \\ \Rightarrow & C' = \frac{1}{3} \\ x = 0 : & 1 = A(-1)^2 + B(-1)(2) + C'(2) \\ \Rightarrow & = \frac{1}{9} - 2B + \frac{2}{3} \\ \Rightarrow & B = -\frac{1}{9} \\ \Rightarrow & \frac{1}{(x+2)(x-1)^2} = \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x-1)^2} \end{aligned}$$

Example

Write $\frac{3x^2 - 3x - 2}{(x-1)(x-2)}$ as partial fractions

$$\begin{aligned} \frac{3x^2 - 3x - 2}{(x-1)(x-2)} &= \frac{3(x-1)(x-2) + 6x - 8}{(x-1)(x-2)} \\ &= 3 + \frac{6x - 8}{(x-1)(x-2)} \\ \text{Suppose } \frac{6x - 8}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\ \Rightarrow 6x - 8 &= A(x-2) + B(x-1) \\ x = 2: \quad 4 &= B \\ x = 1: \quad -2 &= -A \\ \Rightarrow \frac{3x^2 - 3x - 2}{(x-1)(x-2)} &= 3 + \frac{2}{x-1} + \frac{4}{x-2} \end{aligned}$$

Example

Write $\frac{x^2 + 5x + 7}{(x+1)^3}$ as partial fractions

$$\begin{aligned} \frac{x^2 + 5x + 7}{(x+1)^3} &= \frac{(x+1)^2 + 3x + 7}{(x+1)^3} \\ &= \frac{1}{x+1} + \frac{3x+7}{(x+1)^3} \\ &= \frac{1}{x+1} + \frac{3(x+1)+4}{(x+1)^3} \\ &= \frac{1}{x+1} + \frac{3}{(x+1)^2} + \frac{4}{(x+1)^3} \end{aligned}$$

Example

Write $\frac{1}{x^2 - 2x - 1}$ as partial fractions

$$\begin{aligned} \frac{1}{x^2 - 2x - 1} &= \frac{1}{(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))} \\ &= \frac{A}{x - (1 + \sqrt{2})} + \frac{B}{x - (1 - \sqrt{2})} \\ \Rightarrow & 1 = A(x - (1 - \sqrt{2})) + B(x - (1 + \sqrt{2})) \\ x = 1 - \sqrt{2}: & 1 = B((1 - \sqrt{2}) - (1 + \sqrt{2})) \\ \Rightarrow & 1 = B(-2\sqrt{2}) \\ \Rightarrow & B = -\frac{\sqrt{2}}{4} \\ x = 1 + \sqrt{2}: & 1 = A((1 + \sqrt{2}) - (1 - \sqrt{2})) \\ \Rightarrow & 1 = A(2\sqrt{2}) \\ \Rightarrow & A = \frac{\sqrt{2}}{4} \\ \Rightarrow & \frac{1}{x^2 - 2x - 1} = \frac{\sqrt{2}}{4} \left(\frac{1}{x - (1 + \sqrt{2})} - \frac{1}{x - (1 - \sqrt{2})} \right) \end{aligned}$$

Example

Write $\frac{1}{x^3 - 1}$ as partial fractions

$$\begin{aligned} \frac{1}{x^3 - 1} &= \frac{1}{(x - 1)(x^2 + x + 1)} \\ &= \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} \\ \Rightarrow & 1 = A(x^2 + x + 1) + (Bx + C)(x - 1) \\ x = 1: & 1 = 3A \\ \Rightarrow & A = \frac{1}{3} \\ x = 0: & 1 = \frac{1}{3} - C \\ \Rightarrow & C = -\frac{2}{3} \\ x = -1: & 1 = A + (-B + C)(-2) \\ \Rightarrow & 1 = \frac{1}{3} + 2B + \frac{4}{3} \\ \Rightarrow & B = -\frac{1}{3} \\ \Rightarrow & \frac{1}{x^3 - 1} = \frac{1}{3(x - 1)} - \frac{x + 2}{3(x^2 + x + 1)} \end{aligned}$$